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K. HARDY

NOTES ON TWO GENERALIZATIONS OF ALMOST REALCOMPACT SPACES

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Notes on two generalizations of almost realcompact spaces

by

K. Hardy

ABSTRACT

This paper is concerned with two generalizations of almost realcompact spaces which were introduced by DYKES in 1970, namely a-realcompact and c-realcompact spaces. We present some new results on both classes of spaces and provide a reference for pertinent examples which were lacking hitherto. Some open questions are scattered throughout the paper.

KEY WORDS & PHRASES: *almost realcompact spaces, a-realcompact spaces, c-realcompact spaces.*

1. INTRODUCTION

In 1961, FROLÍK [7] (and later [8] for Hausdorff spaces) introduced the class of almost realcompact spaces: a topological space X is called *almost realcompact* if each ultrafilter $\mathcal{U}$ of open subsets of X has the property that $\bigcap cl_X \mathcal{U} = \bigcap\{cl_X U : U \in \mathcal{U}\}$ is non-void whenever $cl_X \mathcal{U}$ has the countable intersection property (cip). The category of Tychonoff almost realcompact spaces (containing the realcompact spaces) is epi-reflective in the category of all Tychonoff spaces; the reflection aX of any Tychonoff space X has recently been constructed explicitly by WOODS [25].

In 1970, DYKES [5] defined the concepts of *a-realcompact* and *c-realcompact* spaces; every regular almost realcompact space is a-realcompact and every Tychonoff almost realcompact space is c-realcompact. We concentrate below on these two generalizations; though categorically less well-behaved than the almost realcompact spaces (both classes are probably not epi-reflective in the category of Tychonoff spaces), their interest lies primarily in their pathology. Our aim here will be to present some new facts, including certain examples which distinguish the two classes (none were provided in [5]) and to indicate some existing open questions.

For convenience, let us recall that a space X is called a *cb-space* (MACK [18]) if, given a decreasing sequence (F_n) of closed subsets of X with empty intersection, there exists a sequence (Z_n) of zero sets with empty intersection and $F_n \subseteq Z_n$ for $n \geq 1$. The cb property is stronger than countable paracompactness and equivalent to it for normal spaces. If the sets (F_n) are all regular closed in X, we say that X is a *weak cb-space* (MACK and JOHNSON [19]).

2. A-REALCOMPACT SPACES

All separation axioms in this section will be stated explicitly as required. DYKES [5, p.573] has defined a topological space X to be *a-realcompact* if every maximal open cover of X has a countable subcover. The notion of a maximal open cover is dual to that of a free ultrafilter of closed sets; thus one has an equivalent formulation: X *is a-realcompact if each ultrafilter of closed subsets of* X *with the cip is fixed*. Spaces with the latter property have appeared in more recent literature ([9], [12], [21], [23]) and have been called *complete with respect to the paving* (see FROLÍK [9] *of all closed subsets* or simply *closed complete spaces*.

It was proved in [5] that a Tychonoff space must be realcompact if it is a-realcompact and a cb-space. An analogue of this result in regular spaces, which extends Theorem 1.6 in [5], is as follows:

PROPOSITION 2.1. *Let* X *be a countably paracompact space. Then* X *is a-realcompact implies* X *is almost realcompact; the converse holds if* X *is regular.*

PROOF. Assume that X is a-realcompact and let $\mathcal{U}$ be an ultrafilter of open subsets of X such that $cl_X\mathcal{U}$ has the cip. Now $cl_X\mathcal{U}$ is a filter base of closed sets and so there is an ultrafilter $\mathcal{F}$ of closed sets containing $cl_X\mathcal{U}$. It follows that $\mathcal{F}$ has the cip: if there exists a decreasing sequence (F_n) in $\mathcal{F}$ with $\cap\{F_n : n \geq 1\}$ void, there is, by the characterization of countable paracompactness due to ISHIKAWA [15], a sequence (V_n) of open sets with $F_n \subseteq V_n$ and $\cap\{cl_X V_n : n \geq 1\}$ void; however, the maximality of $\mathcal{U}$ implies that $V_n \in \mathcal{U}$ for each n, and so $cl_X V_n \in cl_X\mathcal{U}$, contradicting the cip. Finally, $\cap\{F: F \in \mathcal{F}\} \subseteq \cap\{A: A \in cl_X\mathcal{U}\}$ are both non-void. The converse is Theorem 1.6 in [5]. □

BACON [1, p.589] calls a space *isocompact* if each closed and countably compact subset is compact. We generalize Theorem 2.13 in [1] by the following simple observation.

PROPOSITION 2.2. *Every a-realcompact space is isocompact.*

PROOF. Closed subspaces of a-realcompact spaces are a-realcompact and countably compact, a-realcompact spaces must be compact (since countably compact implies cb). □

COROLLARY 2.3. *Let* X *be a* T_4 *a-realcompact space. Then no space* Y *with* $X \subset Y \subseteq \upsilon X$ *can be a k-space.*

PROOF. This is a special case of Theorem 1.1 in [16] which applies to any T_4 isocompact space. □

It follows from Proposition 2.1 that a T_4 a-realcompact space which is *not* almost realcompact must be a Dowker space. Indeed, SIMON [23] has proved that the Dowker space constructed by RUDIN [22], hereafter denoted by R, is a-realcompact but not almost realcompact. Also note that R is a source for many examples of a-realcompact, non-almost realcompact spaces: $R \times I$ is not T_4 and not countably paracompact; the absolute $E(R)$ is extremally disconnected (and incidentally *not* T_4, this result is due to E.K. VAN DOUWEN).

We turn to some covering properties which are closely related with a-realcompact spaces. A space X is *weakly Θ-refinable* if every open cover of X has an open refinement $\mathcal{V} = \cup\{\mathcal{V}_n : n \geq 1\}$ with the property that for each point $x \in X$ there exists a positive integer $n(x)$ such that x meets only finitely many members of $\mathcal{V}_{n(x)}$. If, additionally, each $\mathcal{V}_n$ is a *cover* of X then X is called *Θ-refinable*. We refer to BENNETT and LUTZER [2] for facts on these spaces. In particular, the Θ-refinable spaces include all paracompact and all metacompact spaces.

The symbol (*) will denote the condition that every discrete subspace of a space X is of nonmeasurable cardinality. Thus cl(*) denotes the same condition applied to *closed* discrete subspaces. We will now prove

PROPOSITION 2.4. *A weakly Θ-refinable space is a-realcompact provided* (*) *holds.*

PROOF. Let $\mathcal{F}$ be an ultrafilter of closed subsets of the weakly Θ-refinable space X and assume that $\mathcal{F}$ has the cip while $\cap\{F : F \in \mathcal{F}\}$ is void. Then $\{X\backslash F : F \in \mathcal{F}\}$ is an open cover of X and thus has a weak Θ-refinement $\mathcal{V} = \cup\{\mathcal{V}_n : n \geq 1\}$; actually, $\mathcal{V}$ is a subcover since $\mathcal{F}$ is a filter. For each $n \geq 1$, define $H_{nj} = \{x \in X : x$ is contained in at most j distinct members of $\mathcal{V}_n\}$. We have $H_{nj} \subseteq H_{n,j+1}$ for each $j \geq 1$. Let $X_n = \cup\{H_{nj} : j \geq 1\}$ and $Y_n = \cup\{V : V \in \mathcal{V}_n\}$. Then, $X_n \subseteq Y_n \subseteq X$ and it is easily checked that each H_{nj} is closed in Y_n. Next, it follows that

(i) there are fixed positive integers n,j such that $F \cap H_{nj}$ is non-void for every $F \in \mathcal{F}$: if not, there is for each pair (n,j) a set $F_{nj} \in \mathcal{F}$ with $F_{nj} \cap H_{nj} = \emptyset$; however $X = \cup\{X_n : n \geq 1\}$ and so $\cap\{F_{nj} : n,j \geq 1\}$ is void, contradicting the cip in $\mathcal{F}$.

We now fix attention on the set $H = H_{nj}$ established in (i). Since $\mathcal{V}_n$ is an open cover of H we apply a result of R.L. MOORE (stated nicely as Lemma 2 in [26, p.827]; see also Lemma 8.9 in [3]) to show that

(ii) a discrete subset $D \subseteq H$ exists with the following properties:

(a) $\{st(x,\mathcal{V}_n) : x \in D\}$ covers H

(b) No member of $\mathcal{V}_n$ contains two points of D.

Observe that D is closed in H if X is T_1 and that D is closed (and discrete) in X if X is T_1 and Θ-refinable. Also note that D is uncountable: if $D = \{x_n : n \geq 1\}$, let $V_n = \cup\{V : V \in st\ (x_n,\mathcal{V}_n)\}$; then for $n \geq 1$, $F_n = X\backslash V_n$ is in $\mathcal{F}$, as is $F = \cap_n F_n$ yet $F \cap H$, contradicting (i).

We continue the proof by defining, for each set $F \in \mathcal{F}$, the set $F^* = \{x \in D : st(x,\mathcal{V}_n) \cap F \cap H$ is non-void$\}$ and letting $\mathcal{M} = \{F^* : F \in \mathcal{F}\}$. Then $\mathcal{M}$ is a free filter base on D : if $F \in \mathcal{F}$, take $z \in F \cap H$ and then by (ii)(a) $z \in st(x,\mathcal{V}_n)$ for some $x \in D$ implies $x \in F^*$; that $(F_1 \cap F_2)^* \subseteq F_1^* \cap F_2^*$ is clear; and if $z \in D$, let $st(z,\mathcal{V}_n) = \cup\{V_p : 1 \leq p \leq k, k \leq j\}$, then $z \notin F^*$ for $F = \cap\{X\backslash V_p : 1 \leq p \leq k\}$ and this shows that $\mathcal{M}$ is free. An application of ZORN's Lemma now yields a free ultrafilter $\mathcal{K} \supseteq \mathcal{M}$ on D. By (*), D is realcompact and so there exists a sequence (K_i) in $\mathcal{K}$ with $\cap\{K_i : i \geq 1\}$ void.

For the concluding arguments, define the open sets $U_i = \cup\{st(x,\mathcal{V}_n) : x \in K_i\}$ and notice that $H \cap (\cap\{U_i : i \geq 1\})$ is void: choose $z \in H$ and note that there are $k \leq j$ distinct open sets in $\mathcal{V}_n$ meeting z so that by (ii)(b) we have $J = \{x \in D : z \in st(x,\mathcal{V}_n)\}$ is finite; then $J \cap K_i$ is void for some i and $z \notin U_i$. Also, none of the sets $F_i = X\backslash U_i$ is in $\mathcal{F}$: $x \in K_i$ implies $st(x,\mathcal{V}_n) \cap F_i \cap H$ is void, so that $F_i^* \cap K_i$ is void, contradicting the fact that $\mathcal{K}$ is a filter.

Finally, since $F_i \notin \mathcal{F}$, $i \geq 1$, the maximality of $\mathcal{F}$ implies the existence of $G_i \in \mathcal{F}$ with $G_i \subseteq U_i$ for each $i \geq 1$; but then $G = \cap\{G_i : i \geq 1\}$ is in $\mathcal{F}$ while $G \cap H$ is void, contradicting (i) again. This completes the proof. □

It should be emphasized that the proof of Proposition 2.4 owes much to the ideas found in ZENOR [26] (and later [27]). GARDNER [10] has recently obtained an implicit version (Corollary to Theorem 3.5 together with Theorem 3.9) of Proposition 2.4 under the more restrictive condition that all discrete subspaces are of measure zero; his proofs depend on consideration of regular Borel measures. We pause to state Theorem 3.5 of [10] since it provides an interesting measure-theoretic characterization of a-realcompact spaces: X *is a-realcompact if and only if every two-valued regular Borel measure on* X *is* τ*-additive* (this result also appears as Theorem 1.1 (ii) of REYNOLDS and RICE [21]; as yet unpublished).

We now give some consequences of Proposition 2.4. The first one is an analogue of the famous Theorem of KATĚTOV [17] that: a (Hausdorff) paracompact space is realcompact provided cl(*) holds.

COROLLARY 2.5. *A* Θ*-refinable* T_1*-space is a-realcompact provided* cl(*) *holds.*

Corollary 2.5 furnishes a proof for the fact, announced in HAGER ET AL [12, p.142], that a weakly paracompact (= metacompact) space is a-realcompact if cl(*) holds.

COROLLARY 2.6. *A (Tychonoff) weakly* Θ*-refinable cb-space is realcompact provided* (*) *holds.*

Corollary 2.6. strengthens Corollary 8.10 of BLAIR [3] since every σ-point-finite open refinement is a weak Θ-refinement.

COROLLARY 2.7. *A weakly* Θ*-refinable, countably paracompact space is almost realcompact provided* (*) *holds.*

REMARKS 2.8. Notice that Θ-refinability alone does not imply almost realcompactness: ISBELL's space Ψ [11, 5I] is θ-refinable and pseudocompact (hence not almost realcompact). We also note that the proof in [27] actually yields the fact that *every* Θ*-refinable* T_4*-space is realcompact if* cl(*) *holds.* Thus, the Dowker space R is not Θ-refinable. Whether R is weakly Θ-refinable is not yet known; in fact we can cite no counterexample to the converse of Proposition 2.4, although we conjecture that such spaces exist.
It is unknown if every metalindelöf space (satisfying a mild cardinality condition such as (*)) is a-realcompact; and more generally if *property L* in [1] implies a-realcompactness.

Closed subspaces of a-realcompact spaces are a-realcompact and all compact spaces are a-realcompact. However, we concur with the doubt expressed in [5, p.573] that productivity may fail in general, though we have shown that many products of a-realcompact spaces with almost realcompact spaces are a-realcompact. The following conjecture appears to be well-known (yet absent from the literature) and its substance will be explored in a later paper.

CONJECTURE 2.8. *a-realcompactness is not finitely productive.*
Thus, one would expect the well-known construction (due originally to HERRLICH and VAN DER SLOT; see Theorem 2.1 in [25]) of a maximal a-realcompact extension to fail. Notwithstanding, we now construct an a-realcompact extension of a Tychonoff space X by mimicing the technique used for the construction of the maximal almost realcompact extension aX in [25]: define

$$\alpha_1 X = \{p \in \beta X : \text{there exists an ultrafilter } \mathcal{F} \text{ of closed subsets of X with the cip and } \{p\} = \bigcap\{cl_{\beta X}F : F \in \mathcal{F}\}\}.$$

Now let $\alpha_n X = \alpha_1(\alpha_{n-1}X)$ and put $\alpha X = \bigcup\{\alpha_n X : n \geq 1\}$. It follows that $\alpha_1 X \subseteq a_1 X$ ($a_1 X$ corresponds to all points in υX which are limits of ultrafilters $\mathcal{U}$ of open sets in X such that $cl_X \mathcal{U}$ has cip; see [25]): if $\mathcal{F}$ is an ultrafilter of closed subsets of X with cip, put $\mathcal{A} = \{X \backslash F : F \text{ closed and } F \notin \mathcal{F}\}$; then $\mathcal{A} \subseteq \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter of open sets and $cl_X \mathcal{U} \subseteq \mathcal{F}$ has the cip; $\mathcal{U}$ and $\mathcal{F}$ converge to the same point. Thus, $X \subseteq \alpha X \subseteq aX \subseteq \upsilon X$. We now prove the following

PROPOSITION 2.9. *The space* αX *is a-realcompact.*

PROOF. Let $\mathcal{F}$ be an ultrafilter of closed subsets of αX with the cip. Then, there exists a positive integer $m \geq 1$ such that $F \cap \alpha_m X$ is non-void for all $F \in \mathcal{F}$, for otherwise $\mathcal{F}$ fails to have the cip. It is readily shown that $\mathcal{F}_m = \{F \cap \alpha_m X : F \in \mathcal{F}\}$ is an ultrafilter of closed subsets of $\alpha_m X$; in fact $\mathcal{F}_m$ has the cip since $\mathcal{F}$ is closed under countable intersections. Thus, $\bigcap\{cl_{\beta X}H : H \in \mathcal{F}_m\} = \{p\}$, where $p \in \alpha_{m+1}X$, and so $\bigcap\{F : F \in \mathcal{F}\}$ contains p also. □

Let us comment that little is known about αX : X is a-realcompact if and only if $X = \alpha_1 X (= \alpha X)$; if X is countably paracompact then $\alpha_1 X = a_1 X$, and if X is cb then $\alpha X = \upsilon X$. And many questions remain: for example

a) Is αX the smallest a-realcompact space between X and υX ?

b) A Tychonoff space is pseudocompact if and only if $aX = \beta X$; for which spaces is $\alpha X = \beta X$?

c) For any Tychonoff, pseudocompact, non-countably compact space X, does βX have a subspace which is not a-realcompact (or even a countably compact, non-compact subspace)? A positive answer would prove the conjecture in [12, p.140] since every Borelcomplete space is a-realcompact.

3. C-REALCOMPACT SPACES

All spaces considered in this section are Tychonoff. DYKES [5, p.576] defines a space X to be *c-realcompact* if for every point $x \in \beta X \backslash X$ there exists a real-valued, normal lower semicontinuous (nlsc) function f on βX such that $f(x) = 0$ while $f > 0$ on X. The definition of nlsc functions and some of their properties may be found in [14]; in particular, it is shown there that every space X has a *c-realcompactification* uX with $X \subseteq uX \subseteq \upsilon X$ and the property that every real-valued, locally bounded nlsc function on X has a unique extension to uX. Every almost realcompact space is c-realcompact [5, p.577] (and [14, p.649]) and every weak cb, c-realcompact space is realcompact [5, p.576].

The following characterization is a sharpened version of Lemma 1.1 in [14].

<u>LEMMA 3.1</u>. *A space* X *is c-realcompact if and only if for each point* $x \in \upsilon X \backslash X$ *there exists a decreasing sequence* (A_n) *of regular closed subsets of* X *such that* $\cap\{A_n : n \geq 1\}$ *is void while* $\cap\{cl_{\upsilon X} A_n : n \geq 1\}$ *contains* x.

<u>PROOF</u>. Observe that for any space X and any point x in $\beta X \backslash \upsilon X$ there exists (see for example [6, p.152]) a continuous (hence nlsc) function f on βX with $f(x) = 0$ while $f > 0$ on υX (hence on X). It follows that the points of $\beta X \backslash \upsilon X$ play no essential role in the definition of c-realcompactness. Moreover, if X is dense in some space T and A is regular closed in X then $cl_T A$ is the unique regular closed subset of T with $A = X \cap cl_T A$. These facts, together with Lemma 1.1 in [14], prove the result. □

The inclusions $X \subseteq uX \subseteq a_1 X \subseteq aX \subseteq \upsilon X$ always hold; and $uX = \upsilon X$ if X is weak cb ([14, p.652]). The following result, absent from [14], shows that the latter equality may hold under other conditions.

PROPOSITION 3.2. *If* $x \in \upsilon X \setminus X$ *has a compact neighbourhood then* $x \in uX$. *In particular,* $uX = \upsilon X$ *when* $\upsilon X \setminus X$ *is locally compact.*

PROOF. Let $x \in \upsilon X \setminus X$ and let U be an open subset of υX with $x \in U$ and $cl_{\upsilon X} U$ compact. Consider any decreasing sequence (A_n) of regular closed subsets of X with $x \in \bigcap\{cl_{\upsilon X} A_n : n \geq 1\}$. Put $V_n = U \cap X \cap int_X A_n$. Now by Theorem 4.1 in [4] we have $cl_X(U \cap X)$ is pseudocompact so that $\bigcap\{cl_X V_n : n \geq 1\}$ is non-void and therefore so is $\bigcap\{A_n : n \geq 1\}$. Thus, $x \in uX$. □

A space X is *almost normal* [24] if disjoint pairs of closed subsets of X, one of which is regular closed, have disjoint open neighbourhoods. Equivalently, X is almost normal if each regular closed subset of X is completely separated from each closed set disjoint from it.

PROPOSITION 3.3. *Let* X *be almost normal. Then* X *is c-realcompact implies* X *is a-realcompact.*

PROOF. Let $\mathcal{F}$ be an ultrafilter of closed subsets of X and assume that $\{x\} = \bigcap\{cl_{\upsilon X} F : F \in \mathcal{F}\}$ for some $x \in \upsilon X \setminus X$. If X is c-realcompact there is, by Lemma 3.1, a decreasing sequence (A_n) of regular closed subsets of X with $\bigcap\{A_n : n \geq 1\}$ void while $\bigcap\{cl_{\upsilon X} A_n : n \geq 1\}$ contains x. It follows that, for each n, $A_n \cap F$ is non-void for every $F \in \mathcal{F}$: if $A_n \cap F$ is void for some $F \in \mathcal{F}$ there is a pair of disjoint zero sets W,Z in X with $A_n \subseteq W$ and $F \subset Z$; however $cl_{\upsilon X} W \cap cl_{\upsilon X} Z$ is void, contradicting the position of x. Now the maximality of $\mathcal{F}$ implies that $A_n \in \mathcal{F}$ for $n \geq 1$ so that $\mathcal{F}$ fails to have the cip. Thus, $X = \alpha_1 X = \alpha X$ and we are through. □

REMARKS 3.4. We first note that the converse of Proposition 3.3 is false; specifically, *a* T_4 *a-realcompact space need not be c-realcompact:* the Dowker space R is T_4 and a-realcompact [23]; in [13] we were able to show that R is in fact a weak cb-space so that $uR = aR = \upsilon R$ and hence R is not c-realcompact (nor almost realcompact). Next, *a c-realcompact space exists which is not a-realcompact:* the space X constructed in [19, p.240] is countably paracompact and not weak cb while $\upsilon X = X \cup \{p\}$ is σ-compact (hence cb); thus $X = uX$ (because $uX = \upsilon X$ implies that X is weak cb [14, p.652]); X is not almost realcompact [25, p.206] and so Proposition 2.1 implies that $\alpha X = \upsilon X$.

Topological operations on c-realcompact spaces remain largely a mystery. The intersections of families of c-realcompact spaces lying between X and βX are c-realcompact [14] but subspaces of c-realcompact spaces which are such (beyond the almost realcompact ones) are elusive. The question of productivity for c-realcompact spaces connects with the well-known relation $\upsilon(X\times Y) = \upsilon X \times \upsilon Y$ as follows:

PROPOSITION 3.5. *Let* X *and* Y *be spaces such that the relation* $\upsilon(X\times Y) = \upsilon X \times \upsilon Y$ *holds. If* X *and* Y *are c-realcompact, so is* $X \times Y$; *the converse holds in case* X *and* Y *have isolated points.*

PROOF. Apply Lemma 3.1. □

Let R be the class of spaces X such that for *every* space Y, $\upsilon(X\times Y) = \upsilon X \times \upsilon Y$. Then $X \in R$ implies X is realcompact [20, p.652] and every locally compact, realcompact space of nonmeasurable power belongs to R [4, p.109]. We now have

COROLLARY 3.6. *If* $X \in R$ *and* Y *is c-realcompact then* $X \times Y$ *is c-realcompact.*

CONJECTURE 3.7. *c-realcompactness is not finitely productive.*

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